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Algebraic structure of Green's ansatz and its q-deformed analogue

T D Palev†

International Centre for Theoretical Physics, Trieste, Italy and
Arnold Sommerfeld Institute for Mathematical Physics, Technical University of Clausthal, 38678 Clausthal-Zellerfeld, Germany

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Abstract. The algebraic structure of Green's ansatz is analysed in such a way that its generalization to the case of q-deformed para-Bose and para-Fermi operators becomes evident. To this end, the underlying Lie (super)algebraic properties of the parastatistics are essentially used.

In his first paper on parastatistics [1], Green developed a technique—Green's ansatz technique [2]—appropriate for constructing new (reducible) representations for any set of para-Bose (pB) or para-Fermi (pF) operators. Although quite clear as a mathematical device, the inner structure of Green's ansatz has remained somehow not completely understood. For example, consider m pairs of pB creation and annihilation operators $b_r^{\pm}(p)$, $r = 1, \ldots, m$ of order p. Then, each such operator is represented by a sum

$$b_r^{\pm}(p) = \sum_{k=1}^{p} b_r^{\pm k} \tag{1}$$

where, for any value of k, the $b_r^{\pm k}$ operators obey the Bose commutation relations (here and throughout $[x, y] = xy - yx$, $\{x, y\} = xy + yx$)

$$[b_r^{-k}, b_s^{k}] = \delta_{rs} \qquad [b_r^{-k}, b_s^{-k}] = [b_r^{k}, b_s^{k}] = 0 \qquad \forall r, s \tag{2}$$

and, for $i \neq j$, all operators anticommute

$$\{b_r^{\xi i}, b_s^{\eta j}\} = 0 \qquad \forall \xi, \eta = \pm, \quad i \neq j, \quad r, s. \tag{3}$$

One natural question that arises in relation to the above construction is why the Bose operators partially commute and partially anticommute. Is there any deeper reason for this? The purpose of the present paper is to answer questions like this and, in fact, to show that Green's ansatz construction (1)–(3) is a very natural method from the point of view of the Lie superalgebra (LS).

Much of the motivation for the present work stems from the recent interest in deformed pB and pF operators from various points of view: deformed paraoscillators [3–10] and,

† Permanent address: Institute for Nuclear Research and Nuclear Energy, Boul. Tsarigradsko Chaussee 72, 1784 Sofia, Bulgaria (E-mail palev@bgearn.bitnet).

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more generally, deformed oscillators (see also [11, 12] and references therein in this respect), supersingleton Fock representations of $U_q[osp(1/4)]$ and its singleton structure [13], integrable systems [14–18] and q-parasuperalgebras [19].

In the applications mentioned above, one of the questions is how to construct representations of the deformed paraoperators. In the non-deformed case, Green's ansatz gives, in principle, an answer to this question. Therefore, it is natural to try to extend the same technique to the deformed case. In the present paper, we will analyse the algebraic structure of Green's ansatz in such a way that its generalization to the quantum case will become evident. To this end, we essentially use the circumstance (see also corollary 1) that any n pairs $F_1^\pm, \ldots, F_n^\pm$ of pF operators generate the simple Lie algebra $so(2n+1)$ [20, 21], whereas m pairs of pB operators $B_1^\pm, \ldots, B_m^\pm$ generate an LS [22], which is isomorphic to the basic LS $osp(1/2m)$ [23], denoted also as $B(0/m)$ [24].

In order to be slightly more general and to treat the pB and the pF operators simultaneously, denote a $2(m+n)$-dimensional $\mathbb{Z}_2$-graded linear space ($\mathbb{Z}_2 \equiv (0, 1)$) by $G(n/m)$ with a basis as follows:

$$\text{even basis vectors } C_j^\pm(0) \equiv F_j^\pm \qquad j = 1, \ldots, n \tag{4}$$

$$\text{odd basis vectors } C_i^\pm(1) \equiv B_i^\pm \qquad i = 1, \ldots, m. \tag{5}$$

Let $U(n/m)$ be the free associative unital (= to unity) superalgebra with generators (4) and (5), grading induced from the grading of the generators, and relations

$$[\![[\![C_i^\xi(\alpha), C_j^\eta(\beta)]\!], C_k^\varepsilon(\gamma)]\!] = 2\varepsilon^\gamma \delta_{\beta\gamma}\delta_{jk}\delta_{\varepsilon,-\eta}C_i^\xi(\alpha) - 2\varepsilon^\gamma(-1)^{\beta\gamma}\delta_{\alpha\gamma}\delta_{ik}\delta_{\varepsilon,-\xi}C_j^\eta(\beta) \tag{6}$$

where $\xi, \eta, \varepsilon = \pm$, $\alpha, \beta, \gamma \in \mathbb{Z}_2$ and i, j, k take all possible values according to (4) and (5). In (6) and throughout, $[\![\ ,\ ]\!]$ is a supercommutator, defined on any two homogeneous elements a, b from $U(n/m)$ as

$$[\![a, b]\!] = ab - (-1)^{\deg(a)\deg(b)}ba. \tag{7}$$

In the case $\alpha = \beta = \gamma = 0$, equation (6) reduces to

$$[[F_i^\xi, F_j^\eta], F_k^\varepsilon] = 2\delta_{jk}\delta_{\varepsilon,-\eta}F_i^\xi - 2\delta_{ik}\delta_{\varepsilon,-\xi}F_j^\eta \tag{8}$$

whereas, for $\alpha = \beta = \gamma = 1$, it gives

$$[\{B_i^\xi, B_j^\eta\}, B_k^\varepsilon] = 2\varepsilon\delta_{jk}\delta_{\varepsilon,-\eta}B_i^\xi + 2\varepsilon\delta_{ik}\delta_{\varepsilon,-\xi}B_j^\eta. \tag{9}$$

Equations (8) and (9) are the defining relations for the pF and pB operators, respectively [1].

Relations (6) define a structure of a Lie-super triple system [25] on $G(n/m) \subset U(n/m)$ with a triple product $G(n/m) \otimes G(n/m) \otimes G(n/m) \to G(n/m)$ defined as

$$[\![[\![x, y]\!], z]\!] = 2\langle y|z\rangle x - 2(-1)^{\deg(x)\deg(y)}\langle x|z\rangle y \in G(n/m) \qquad \forall x, y, z \in G(n/m) \tag{10}$$

where the bilinear form $\langle x|y\rangle$ is defined in agreement with (6) to be [25]

$$\langle C_i^\xi(\alpha)|C_j^\eta(\beta)\rangle = \eta^\alpha\delta_{ij}\delta_{\alpha\beta}\delta_{\xi,-\eta} \qquad \xi, \eta = \pm \qquad \alpha, \beta \in \mathbb{Z}_2. \tag{11}$$

Consider $U(n/m)$ as an LS with a supercommutator (7). Then it is straightforward to check that (lin.env. = linear envelope)

$$B(n/m) = \text{lin.env.}\{[\![C_i^\xi(\alpha), C_j^\eta(\beta)]\!],\ C_k^\varepsilon(\gamma)|\forall i, j, k,\ \xi, \eta, \varepsilon = \pm,\ \alpha, \beta, \gamma \in \mathbb{Z}_2\} \tag{12}$$

is a subalgebra of the LS $U(n/m)$.

Proposition 1 [26]. The LS $B(n/m)$ is isomorphic to the orthosymplectic LS $osp(2n+1/2m)$. The associative superalgebra $U(n/m)$ is its universal enveloping algebra $U[osp(2n+1/2m)]$.

The first part of the proposition was proved in [26]. The second part follows from the following two observations:

(i) the supercommutation relations between all generators of $osp(2n+1/2m)$ (which constitute a basis in the underlying linear space) follow from relations (6) between only the Lie-super triple generators (4) and (5); and

(ii) the universal enveloping algebra of a given LS is the free associative unital algebra of its generators and the supercommutation relations they satisfy.

Observe that everywhere in the above considerations, the pF operators appear as even (i.e. bosonic) variables, whereas the pB are odd (i.e. fermionic) operators. Moreover, the pB do not commute with the pF. Okubo [25] and Macfarlane [15] have also recently arrived at this same conclusion.

As an immediate consequence of the above proposition, we have the following corollary.

Corollary 1.

(i) [20, 21] The free associative unital algebra of the pF operators (4) is isomorphic to the universal enveloping algebra $U[so(2n+1)]$ of the orthogonal Lie algebra $so(2n+1)$:

$$so(2n+1) = \text{lin.env.}\{[F_i^\xi, F_j^\eta], F_k^\varepsilon | i, j, k = 1, \ldots, n;\ \xi, \eta, \varepsilon = \pm\} \subset U[so(2n+1)]. \tag{13}$$

(ii) [23] The free associative unital algebra of the pB operators (5) is isomorphic to the universal enveloping algebra $U[osp(1/2m]$ of the orthosymplectic LS $osp(1/2m)$:

$$osp(1/2m) = \text{lin.env.}\{\{B_i^\xi, B_j^\eta\}, B_k^\varepsilon | i, j, k = 1, \ldots, m;\ \xi, \eta, \varepsilon = \pm\} \subset U[osp(1/2m)]. \tag{14}$$

Corollary 2. The representation theory of the Lie-super triple system $G(n/m)$ with generators (4), (5) and relations (6) is completely equivalent to the representation theory of the orthosymplectic LS $osp(2n+1/2m)$. In particular, the problem of constructing the representations of n pairs of pF operators (4) is equivalent to the problem of constructing the representations of the Lie algebra $so(2n+1)$; similarly, the representation theory of m pairs of pB operators is the same as the representation theory of the LS $osp(1/2m)$.

The finite-dimensional representations of $so(2n+1)$ are known; they have been explicitly constructed [27]. All representations of the pF operators corresponding to a fixed order of the parastatistics are among the finite-dimensional representations. The pF operators have, however, several other representations [28], including representations with degenerate vacua. In a more practical aspect, the results of [27] are, unfortunately, not so useful for the pF statistics. The point is that the transformation relations of the Gel'fand–Zetlin basis in [27] are given for a set of $2n$ operators (generating all the rest of the $2n^2+n$ generators), which are different from the pF operators and also different from the $3n$ Chevalley generators. The relations between the pF operators and the operators used in [27] are not linear.

The above corollaries are not of great practical use for the representations of the pB operators or, more generally, of the Lie-super triple system $G(n/m)$. So far, only the finite-dimensional representations of $osp(2n+1/2m)$ have been classified [29]. Explicit expressions for the matrix elements are available only for low-rank algebras (see [30] and references therein). Moreover, the interesting representations of the pB operators are infinite-dimensional. The Lie-super triple system $G(n/m)$ and, hence, $osp(2n+1/2m)$ have, however, one simple but important representation: the Fock representation, which is of particular interest for our considerations.

Proposition 2 [26]. Denote by $W(n/m)$ the antisymmetric Clifford–Weyl superalgebra, namely, the associative algebra generated by n pairs of Fermi creation and annihilation operators (CAOs) $f_i^{\pm} \equiv c_i^{\pm}(0)$, $i, j = 1, \ldots, n$

$$\{f_i^{\xi}, f_j^{\eta}\} \equiv \{c_i^{\xi}(0), c_j^{\eta}(0)\} = \delta_{ij}\delta_{\xi,-\eta} \qquad \xi, \eta = \pm \tag{15}$$

and m pairs of Bose CAOs $b_j^{\pm} \equiv c_j^{\pm}(1)$, $i, j = 1, \ldots, m$

$$[b_i^{\xi}, b_j^{\eta}] \equiv [c_i^{\xi}(1), c_j^{\eta}(1)] = \eta\delta_{ij}\delta_{\xi,-\eta} \qquad \xi, \eta = \pm \tag{16}$$

under the condition that the Bose operators anticommute with the Fermi operators

$$\{f_i^{\xi}, b_j^{\eta}\} \equiv \{c_i^{\xi}(0), c_j^{\eta}(1)\} = 0 \qquad \xi, \eta = \pm \qquad i = 1, \ldots, n \qquad j = 1, \ldots, m. \tag{17}$$

$W(n/m)$ is an associative superalgebra with grading induced from the requirement that the Fermi operators are even elements and the Bose operators are odd. Consider $W(n/m)$ as an algebra of (linear) operators in the corresponding Fock space $H \equiv H(n/m)$, $W(n/m) \subset \mathrm{End}(H)$. Then, the map

$$\pi : osp(2n+1/2m) \to W(n/m) \qquad \text{defined as } \pi(C_i^{\xi}(\alpha)) = c_i^{\xi}(\alpha) \quad \forall \xi = \pm \text{ and } i \tag{18}$$

is a Fock representation of $osp(2n+1/2m)$ or a representation of the Lie-super triple system $G(n/m)$, which are the same.

In order to prove the proposition, one has simply to check that relations (6) remain valid after the replacement $C_i^{\xi}(\alpha) \to c_i^{\xi}(\alpha)$.

In the case of $so(2n+1)$, or equivalently of n pairs of pF operators (resp. of $osp(1/2m)$ or equivalently of m pairs of pB operators), proposition 2 reduces to the usual representation of the pF operators with Fermi operators (resp. of the pB operators with Bose operators).

The conclusion, relevant for us, is that operators (4) and (5) generate an associative superalgebra, namely, $U[osp(2n+1/2m)]$ (proposition 1) and that we know at least one representation of $U[osp(2n+1/2m)]$, namely its Fock representation (proposition 2).

For simplicity, set $L = osp(2n+1/2m)$, $U = U[osp(2n+1/2m)]$ and let $L^{\otimes p}$ and $U^{\otimes p}$ be their pth tensorial powers. Introduce the following notation (e is the unity of U):

$$L^k = \{e_1 \otimes \ldots e_{k-1} \otimes a \otimes e_{k+1} \otimes \ldots \otimes e_p | a \in L \ e_i = e \forall i \neq k\}. \tag{19}$$

$$U^k = \{e_1 \otimes \ldots e_{k-1} \otimes u \otimes e_{k+1} \otimes \ldots \otimes e_p | u \in U \ e_i = e \forall i \neq k\}. \tag{20}$$

Then, the map $\tau^k : L \to L^k \subset U^k$

$$\tau^k(a) = e_1 \otimes \ldots e_{k-1} \otimes a \otimes e_{k+1} \otimes \ldots \otimes e_p \qquad a \in L \ e_i = e \forall i \neq k \tag{21}$$

is an LS morphism of L onto L^k; the same map (21) considered for all $a \in U$ is an associative algebra morphism of U onto U^k.

The set of elements (21) generate U^k and, since

$$U^{\otimes p} = U^1 U^2 \ldots U^p \tag{22}$$

the elements (21) considered for all $k = 1, \ldots, p$ generate $U^{\otimes p}$.

The sum

$$\Delta^{(p)} = \tau^1 + \tau^2 + \cdots + \tau^p \ : \ L \longrightarrow L^{\otimes p} \tag{23}$$

is an LS morphism, the 'diagonal' LS morphism, of L into $L^{\otimes p}$, which is extended to a morphism of the associative algebra U into the associative algebra $U^{\otimes p}$ in a natural way

$$\Delta^{(p)}(a_1 a_2 \ldots a_m) = \Delta^{(p)}(a_1)\Delta^{(p)}(a_2)\ldots\Delta^{(p)}(a_m) \qquad \forall\, a_1, a_2, \ldots, a_m \in L. \tag{24}$$

Let $\pi^1, \pi^2, \ldots, \pi^p$ be (not necessarily different) representations of $L = osp(2n+1/2m)$ (and, hence, of $U = U[osp(2n+1/2m)]$) in the $\mathbb{Z}_2$-graded linear spaces $H^1, H^2, \ldots, H^p$, respectively, i.e. the operators $\pi^k[C_r^{\pm}(\alpha)] \in \mathrm{End}(H^k)$, $k = 1, \ldots, p$ satisfy the Lie-super triple relations (6) and

$$\deg\{\pi^k[C_r^{\pm}(\alpha)]\} = \alpha. \tag{25}$$

Then

$$\pi^1 \otimes \pi^2 \otimes \ldots \otimes \pi^p : U^{\otimes p} \longrightarrow \mathrm{End}(H^1 \otimes H^2 \otimes \ldots \otimes H^p) \tag{26}$$

gives a representation of both the LS $\Delta^{\otimes p}(L)$ and of the associative algebra $U^{\otimes p}$. The composition maps ($k = 1, \ldots, p$)

$$(\pi^1 \otimes \pi^2 \otimes \ldots \otimes \pi^p) \circ \tau^k : U[osp(2n+1/2m)] \longrightarrow \mathrm{End}(H^1 \otimes H^2 \otimes \ldots \otimes H^p) \tag{27}$$

$$(\pi^1 \otimes \pi^2 \otimes \ldots \otimes \pi^p) \circ \Delta^{(p)} : U[osp(2n+1/2m] \longrightarrow \mathrm{End}(H^1 \otimes H^2 \otimes \ldots \otimes H^p) \tag{28}$$

give representations of both the LS $osp(2n+1/2m)$ and the associative algebra $U[osp(2n+1/2m)]$. Therefore, the operators

$$\begin{aligned}\hat{c}_r^{\pm k}(\alpha) &= [(\pi^1 \otimes \pi^2 \otimes \ldots \otimes \pi^p) \circ \tau^k] C_r^{\pm}(\alpha) \\ &= id^1 \otimes \ldots \otimes id^{k-1} \otimes \pi^k[C_r^{\pm}(\alpha)] \otimes id^{k+1} \otimes \ldots \otimes id^p \in \mathrm{End}(H^1 \otimes H^2 \otimes \ldots \otimes H^p) \\ &\quad k = 1, \ldots, p \qquad \alpha = \pm\end{aligned} \tag{29}$$

and

$$\hat{c}_r^{\pm}(p, \alpha) = [(\pi^1 \otimes \pi^2 \otimes \ldots \otimes \pi^p) \circ \Delta^{(p)}] C_r^{\pm}(\alpha) = \sum_{k=1}^{p} \hat{c}_r^{\pm k}(\alpha) \tag{30}$$

satisfy the Lie-super triple relations (6). From the very definition of a tensor product of associative algebras [31], we obtain (for all r, s according to (4) and (5) and $\xi, \eta = \pm$)

$$[\![\hat{c}_r^{\xi i}(\alpha), \hat{c}_s^{\eta j}(\beta)]\!] \equiv \hat{c}_r^{\xi i}(\alpha)\hat{c}_s^{\eta j}(\beta) - (-1)^{\alpha\beta}\hat{c}_s^{\eta j}(\beta)\hat{c}_r^{\xi i}(\alpha) = 0 \qquad i \neq j = 1, \ldots, p. \tag{31}$$

In particular, for $\alpha = 1$,

$$\hat{b}_r^{\pm k} \equiv \hat{c}_r^{\pm k}(1) = id^1 \otimes \ldots \otimes id^{k-1} \otimes \pi^k(B_r^{\pm}) \otimes id^{k+1} \otimes \ldots \otimes id^p \tag{32}$$

$$\hat{b}_r^{\pm}(p) \equiv \hat{c}_r^{\pm}(p, 1) = \sum_{k=1}^{p} \hat{b}_r^{\pm k} \qquad r = 1, \ldots, m \tag{33}$$

and

$$\{\hat{b}_r^{\xi i}, \hat{b}_s^{\eta j}\} = 0 \qquad i \neq j = 1, \ldots, p, \quad r, s = 1, \ldots, m, \quad \xi, \eta = \pm \tag{34}$$

whereas the operators $\hat{b}_r^{\pm k}$ with the same upper-case index k satisfy the pB relations (9) and maybe also, particularly for the representation π^k, other relations.

Similarly, for $\alpha = 0$,

$$\hat{f}_r^{\pm k} \equiv \hat{c}_r^{\pm k}(0) = id^1 \otimes \ldots \otimes id^{k-1} \otimes \pi^k(F_r^\pm) \otimes id^{k+1} \otimes \ldots \otimes id^p \tag{35}$$

$$\hat{f}_r^\pm(p) \equiv \hat{c}_r^\pm(p, 0) = \sum_{k=1}^{p} \hat{f}_r^{\pm k} \qquad r = 1, \ldots, n \tag{36}$$

and

$$[\hat{f}_r^{\xi i}, \hat{f}_s^{\eta j}] = 0 \qquad i \neq j = 1, \ldots, p, \quad r, s = 1, \ldots, n, \quad \xi, \eta = \pm. \tag{37}$$

Consider now the important case when all representations $\pi^1, \pi^2, \ldots, \pi^p$ are the same and coincide with the Fock representation, namely, π is a morphism of $U[osp(2n+1/2m)]$ onto the Clifford–Weyl algebra $W(n/m)$ defined in (18)

$$\pi^1 = \pi^2 = \cdots = \pi^p = \pi. \tag{38}$$

In order to distinguish this particular case, we do not write any more hats over the operators. Then, from (30), we obtain

$$c_r^\pm(p, \alpha) = [\pi^{\otimes p} \circ \Delta^{(p)}] C_r^\pm(\alpha) = \sum_{k=1}^{p} c_r^{\pm k}(\alpha) \in \mathrm{End}(H^{\otimes p}) \tag{39}$$

where according to (29) and (18)

$$c_r^{\pm k}(\alpha) = id^1 \otimes \ldots \otimes id^{k-1} \otimes c_r^\pm(\alpha) \otimes id^{k+1} \otimes \ldots \otimes id^p \qquad k = 1, \ldots, p \qquad \alpha = \pm \tag{40}$$

and the operators $c_r^{\pm k}(\alpha)$ satisfy, according to (15)–(17) and (31), the relations (see (7))

$$[\![c_r^{\xi i}(\alpha), c_s^{\eta j}(\beta)]\!] = \eta^\alpha \delta_{rs} \delta_{ij} \delta_{\alpha\beta} \delta_{\xi,-\eta} \qquad \xi, \eta = \pm \qquad \alpha, \beta \in \mathbb{Z}_2. \tag{41}$$

Setting $b_r^\pm(p) = c_r^\pm(p, 1)$, we obtain from (39) and (41) Green's ansatz for the pB operators of order p

$$b_r^\pm(p) = [\pi^{\otimes p} \circ \Delta^{(p)}] B_r^\pm = \sum_{k=1}^{p} b_r^{\pm k} \tag{42}$$

where

$$b_r^{\pm k} = id^1 \otimes \ldots \otimes id^{k-1} \otimes b_r^\pm \otimes id^{k+1} \otimes \ldots \otimes id^p \qquad k = 1, \ldots, p \qquad r = 1, \ldots, m. \tag{43}$$

As follows from (40) (or immediately from (43), taking into account that $b_r^{\pm}$ are odd operators), the Bose operators $b_r^{\pm k}$ partially commute and partially anticommute. More precisely,

$$[b_r^{-k}, b_s^k] = \delta_{rs} \qquad [b_r^{-k}, b_s^{-k}] = [b_r^k, b_s^k] = 0 \qquad \forall k, r, s \tag{44}$$

and, for $i \neq j$, all operators anticommute

$$\{b_r^{\xi i}, b_s^{\eta j}\} = 0 \qquad \xi, \eta = \pm \qquad i \neq j. \tag{45}$$

Similarly, setting $f_r^{\pm}(p) = c_r^{\pm}(p, 0)$, we obtain, from (39) and (41), Green's ansatz for the pF operators of order p

$$f_r^{\pm}(p) = [\pi^{\otimes p} \circ \Delta^{(p)}]F_r^{\pm} = \sum_{k=1}^{p} f_r^{\pm k} \tag{46}$$

where

$$f_r^{\pm k} = id^1 \otimes \ldots \otimes id^{k-1} \otimes f_r^{\pm} \otimes id^{k+1} \otimes \ldots \otimes id^p \qquad k = 1, \ldots, p \qquad r = 1, \ldots, n. \tag{47}$$

Setting $\alpha = 0$ in (40) (or directly from (47), taking into account that $f_r^{\pm}$ are even operators), one obtains

$$\{f_r^{-k}, f_s^k\} = \delta_{rs} \qquad \{f_r^{-k}, f_s^{-k}\} = \{f_r^k, f_s^k\} = 0 \qquad \forall k, r, s \tag{48}$$

and for $i \neq j$ all operators commute

$$[f_r^{\xi i}, f_s^{\eta j}] = 0 \qquad \xi, \eta = \pm \qquad i \neq j. \tag{49}$$

From the above considerations, it is clear that Green's ansatz representation (46) of the pF operators of order p is simply given as a representation of the pF operators $f_r^{\pm}(p)$, considered as generators of the universal enveloping algebra $U[so(2n+1)]$) in the tensor product of p copies of (irreducible, finite-dimensional) Fock representations of the Lie algebra $so(2n+1)$.

Similarly, Green's ansatz representation (42) of the pB operators of order p gives a representation of the pB operators in the tensor product of p copies of (irreducible, infinite-dimensional) Fock representations of the LS $osp(1/2m)$.

Equations (39) and (41) generalize the concept of Green's ansatz to the case of Lie-super triple operators (4) and (5), which are free generators, with relations (6), of the universal enveloping algebra $U[osp(2n+1/2m)]$. The representation of the generators $C_r^{\pm}(\alpha)$ (= the representation of $osp(2n+1/2m)$) is realized in the tensor product space $H^{\otimes p}$ of p copies of Fock representations (18) of $osp(2n+1/2m)$. Therefore, Green's ansatz gives a highly reducible representation of the Lie-super triple operators. In particular, this is the case if only pB or pF operators are present. If $|0\rangle \in H$ is the highest-weight vector in H, then the irreducible subspace, containing $|0\rangle^{\otimes p} \in H^{\otimes p}$, carries a representation corresponding to an order of statistics p. The other irreducible components of $H^{\otimes p}$ also contain vacuum-like states and among these are the highest-weight vectors. The corresponding representations, however, no longer correspond to representations with a fixed statistical order, namely to representations with unique vacuum states (see, for example, [28]). The problem to decompose $H^{\otimes p}$ into a direct sum of irreducible subspaces with respect to the para-operators (= with respect to $osp(2n+1/2m)$) or even the simpler problem—to extract the irreducible submodule carrying only the representation with a statistical order p—has not been solved so far. The problem is also not solved for the case of only pF ($m = 0$) or pB ($n = 0$) operators.

Passing to a discussion of a possible generalization of Green's ansatz (39) to the case of deformed operators, we first observe that, in all cases, Green's ansatz is obtained (see (39), (42) and (46)) as a two-step procedure, namely as a composition of two (associative algebra) morphisms:

$$\begin{aligned} &\Delta^{(p)} : U[osp(2n+1/2m)] \longrightarrow U[osp(2n+1/2m)]^{\otimes p} \\ &\pi^{\otimes p} : U[osp(2n+1/2m)]^{\otimes p} \longrightarrow \mathrm{End}(H^{\otimes p}). \end{aligned} \tag{50}$$

In the following, we consider only (one-parameter) deformations of the Lie-super triple generators (4) and (5) which generate a Hopf deformation $U_q[osp(2n+1/2m)]$ of $U[osp(2n+1/2m)]$. By a Hopf deformation, we mean a deformation of $U[osp(2n+1/2m)]$ which preserves its Hopf algebra structure (as defined, for instance, in [32]).

The deformed version of Green's ansatz of order p will be based on a deformation of relation (39), namely,

$$c_r^{\pm}(p,\alpha) = [\pi^{\otimes p} \circ \Delta^{(p)}]C_r^{\pm}(\alpha). \tag{51}$$

To this end, we need to define deformed versions of the operators $\Delta^{(p)}$ and $\pi^{\otimes p}$ so that they remain morphisms and deformed Lie-super triple generators $C_r^{\pm}(\alpha)_q \equiv C_r^{\pm}(\alpha)$ (whenever possible, we suppress the subscript q for the deformed objects).

In order to define a deformed analogue of the operator $\Delta^{(p)}$, we use the circumstance that the superalgebra $U = U[osp(2n+1/2m]$ is a Hopf superalgebra with a comultiplication Δ, defined as

$$\Delta(a) = a\otimes e + e\otimes a \qquad \forall a \in L \qquad \Delta(e) = e\otimes e. \tag{52}$$

From (21) and (23), we deduce that

$$\Delta^{(2)} = \Delta \qquad \Delta^{(3)} = (id\otimes\Delta)\circ\Delta^{(2)} \qquad \Delta^{(k)} = [(id^{\otimes(k-2)})\otimes\Delta]\circ\Delta^{(k-1)}. \tag{53}$$

The important point is that the operators $\Delta^{(k)}$ also preserve the property to be morphisms after the quantization (= Hopf deformation) of $U(L)$, i.e. the map

$$\Delta^{(p)} : U_q[osp(2n+1/2m)] \longrightarrow U_q[osp(2n+1/2m)]^{\otimes p} \tag{54}$$

is an associative algebra morphism. Certainly, in the deformed case, equation (52) has to be replaced with the corresponding expression for the comultiplication on $U_q[osp(2n+1/2m)]$.

In order to determine the deformed analogue of the operator $\pi^{\otimes p}$, we observe that if $\pi : U_q[osp(2n+1/2m)] \to \mathrm{End}(H)$ is a representation of $U_q[osp(2n+1/2m)]$ in the linear space H, then

$$\pi^{\otimes p} : U_q[osp(2n+1/2m)]^{\otimes p} \longrightarrow \mathrm{End}(H^{\otimes p}) \tag{55}$$

is a representation of $U_q[osp(2n+1/2m)]^{\otimes p}$. In the non-deformed case, π is a morphism of $U[osp(2n+1/2m)$ onto the Clifford–Weyl algebra $W(n/m)$ (see (18)). Therefore, it is natural to assume that, in the deformed case, π is a morphism of $U_q[osp(2n+1/2m)]$ onto a deformed algebra $W_q(n/m)$. We recall the definition [33].

Definition 1. The deformed antisymmetric Clifford–Weyl superalgebra $W_q(n/m)$ is an associative unital superalgebra with free generators $f_i^{\pm}, k_i(0) \equiv k_i^{+}(0), k_i^{-}(0),\ i = 1, \ldots, n$ and $b_j^{\pm}, k_j(1) \equiv k_j^{+}(1), k_j^{-}(1),\ j = 1, \ldots, m$, which obey the relations

$$k_i(0)k_j(0) = k_j(0)k_i(0) \qquad k_i(0)k_i^{-}(0) = 1 \qquad k_i(0)f_j^{\varepsilon} = q^{\varepsilon\delta_{ij}} f_j^{\varepsilon} k_i(0) \tag{56a}$$

$$f_i^{\varepsilon} f_j^{\varepsilon} + f_j^{\varepsilon} f_i^{\varepsilon} = 0 \qquad f_i^{-} f_j^{+} + q^{2\varepsilon\delta_{ij}} f_j^{+} f_i^{-} = \delta_{ij}(k_i(0)^{\varepsilon})^2$$
$$j = 1, \ldots, n \qquad \varepsilon = \pm \tag{56b}$$

$$k_i(1)k_j(1) = k_j(1)k_i(1) \qquad k_i(1)k_i^{-}(1) = 1 \qquad k_i(1)b_j^{\varepsilon} = q^{\varepsilon\delta_{ij}} b_j^{\varepsilon} k_i(1) \tag{57a}$$

$$b_i^{\varepsilon} b_j^{\varepsilon} - b_j^{\varepsilon} b_i^{\varepsilon} = 0 \qquad b_i^{-} b_j^{+} - q^{2\varepsilon\delta_{ij}} b_j^{+} b_i^{-} = \delta_{ij}(k_i(1)^{-\varepsilon})^2$$
$$i, j = 1, \ldots, m \qquad \varepsilon = \pm \tag{57b}$$

$$f_i^{\varepsilon} b_j^{\eta} = -b_j^{\eta} f_i^{\varepsilon} \qquad k_i(0)k_j(1) = k_j(1)k_i(0) \tag{58a}$$

$$k_i(0)b_j^{\varepsilon} = b_j^{\varepsilon} k_i(0) \qquad k_j(1)f_i^{\varepsilon} = f_i^{\varepsilon} k_j(1) \qquad i = 1, \ldots, n$$
$$j = 1, \ldots, m \qquad \varepsilon = \pm. \tag{58b}$$

The grading on $W_q(n/m)$ is induced from the grading of its generators

$$\deg(f_i^{\pm}) = \deg(k_i(0)) = \deg(k_i(1)) = 0 \qquad \deg(b_i^{\pm}) = 1. \tag{59}$$

In the $n = 0$ case, $W_q(0/m)$ is the associative superalgebra, generated by m triples $b_r^{\pm}$, $k_r = q^{N_r}$, $r = 1, \ldots, m$ of commuting deformed Bose operators, as defined in [34–36]. It is described by equations (57) and N_r is the rth boson number operator. In the $m = 0$ case, $W_q(n/0)$ is the associative algebra of $3n$ triples of deformed Fermi operators [37], given by equations (56).

It remains to determine the deformed creation and annihilation operators. This problem is, so far, only partially solved. The definition of m pairs of deformed pB operators $B_i^{\pm}$, $i = 1, \ldots, m$, which together with the 'Cartan' elements $K_1, \ldots, K_m$ generate $U_q[osp(1/2m)]$, was given in [9]. In terms of the Chevalley generators E_i, F_i, K_i, $i = 1, \ldots, m$ they read $(i = 1, \ldots, m-1)$

$$B_i^{-} = -\sqrt{\frac{2q}{(q+q^{-1})}}[E_i, [E_{i+1}, [E_{i+2}, [\ldots, [E_{m-2}, [E_{m-1}, E_m]_{q^{-2}}]_{q^{-2}} \ldots]_{q^{-2}}$$
$$\times K_i K_{i+1} \ldots K_m \tag{60}$$

$$B_m^{-} = -\sqrt{\frac{2q}{(q+q^{-1})}} E_m K_m$$

and

$$B_i^{+} = \sqrt{\frac{2q}{(q+q^{-1})}}[\ldots, [F_m, F_{m-1}]_{q^2}, F_{m-2}]_{q^2}, \ldots]_{q^2}, F_{i+2}]_{q^2}, F_{i+1}]_{q^2}, F_i]_{q^2}$$
$$\times K_i^{-1} K_{i+1}^{-1} \ldots K_m^{-1} \tag{61}$$

$$B_m^{+} = \sqrt{\frac{2q}{(q+q^{-1})}} F_m K_m^{-1}.$$

The morphism π of $U_q[osp(1/2m)]$ onto $W_q(0/m)$, namely the operators $\pi(B_i^{\pm}) \in W_q(0/m)$, were constructed in [9]. In [8] (in somewhat different notation), we have shown that the Chevalley generators can be expressed in terms of the $3n$ pre-oscillator generators $B_i^{\pm}, K_i$, $i = 1, \ldots, m$, thus giving a new definition of $U_q[osp(1/2m)]$ entirely in terms of the pre-oscillator generators. Thus, we are ready to state the following result.

Proposition 3. If π is the Fock representation of $U_q[osp(1/2m)$ [9] (resp. of $U_q[so(2n+1)])$ and $\Delta^{(p)}$ is the nth-order coproduct operator (54), then equation (51) defines the deformed analogue of Green's ansatz of order p for m pairs of deformed pB operators (resp. for n pairs of deformed pF operators).

The proof is evident since the composition of the morphisms $\pi^{\otimes p}$ and $\Delta^{(p)}$ is also a morphism. Hence, the operators $c_r^{\pm}(p, \alpha)_q \equiv c_r^{\pm}(p, \alpha)$, defined by (54), satisfy the same relations as $C_r^{\pm}(\alpha)_q$; in the limit $q \to 1$, they reduce to the corresponding non-deformed pB or pF operators of order p.

Proposition 3 could also be extended to deformed Lie-super triple systems. So far, however, such a deformation has been carried out only for the case $n = m = 1$ [33].

As an example, we consider in more detail Green's ansatz of order $p = 2$ related to $U_q[osp(1/4)]$, namely, the ansatz corresponding to two pairs of deformed pB operators $b_1^{\pm}(2)$ and $b_2^{\pm}(2)$ [39]. To this end, we recall first the definition of $U_q[osp(1/4)]$ in terms of its Chevalley generators. We choose the Cartan matrix (α_{ij}) as in [40], i.e. this is a 2×2 symmetric matrix with

$$\alpha_{11} = 2 \qquad \alpha_{22} = 1 \qquad \alpha_{12} = \alpha_{21} = -1. \tag{62}$$

Then $U_q[osp(1/4)]$ is the free associative superalgebra with Chevalley generators E_i, F_i, K_i, $i = 1, 2$, graded as

$$\deg(E_2) = \deg(F_2) = 1 \qquad \deg(E_1) = \deg(F_1) = \deg(K_1) = \deg(K_2) = 0 \tag{63}$$

which satisfies the Cartan relations

$$K_i K_i^{-1} = K_i^{-1} K_i = 1 \qquad K_i K_j = K_j K_i \qquad i, j = 1, 2, \tag{64}$$

$$K_i E_j = q^{\alpha_{ij}} E_j K_i \qquad K_i F_j = q^{-\alpha_{ij}} F_j K_i \qquad i, j = 1, 2, \tag{65}$$

$$[E_2, F_2] = \frac{K_2^2 - K_2^{-2}}{q - q^{-1}} \qquad [E_i, F_j] = \delta_{ij} \frac{K_i^2 - K_i^{-2}}{q^2 - q^{-2}} \qquad \forall i, j = 1, 2 \text{ except } i = j = 2 \tag{66}$$

the Serre relations for the simple positive root vectors

$$E_1^2 E_2 - (q^2 + q^{-2}) E_1 E_2 E_1 + E_2 E_1^2 = 0 \tag{67}$$

$$E_2^3 E_1 + (1 - q^2 - q^{-2})(E_2^2 E_1 E_2 + E_2 E_1 E_2^2) + E_1 E_2^3 = 0 \tag{68}$$

and the Serre relations obtained from (67)–(68) by replacing everywhere E_i by F_i.

The action of the coproduct $\Delta : U_q \to U_q \otimes U_q$ can be given as $(i = 1, 2)$

$$\Delta(E_i) = E_i \otimes K_i + K^{-1} \otimes E_i \qquad \Delta(F_i) = F_i \otimes K_i + K_i^{-1} \otimes F_i \qquad \Delta(K_i) = K_i \otimes K_i. \tag{69}$$

In the $n = 2$ case, we obtain, from (60) and (61), the expressions for the deformed pB operators

$$B_1^- = -\sqrt{\frac{2q}{(q+q^{-1})}}[E_1, E_2]_{q^{-2}} K_1 K_2 \qquad B_2^- = -\sqrt{\frac{2q}{(q+q^{-1})}} E_2 K_2 \tag{70}$$

and

$$B_1^+ = \sqrt{\frac{2q}{(q+q^{-1})}}[F_2, F_1]_{q^2} K_1^{-1} K_2^{-1} \qquad B_2^+ = \sqrt{\frac{2q}{(q+q^{-1})}} F_2 K_2^{-1}. \tag{71}$$

From (69)–(71), and using the circumstance that the comultiplication is an algebra morphism, we obtain

$$\Delta(B_1^+) = q^{-1/2} B_1^+ \otimes K_1 K_2^{-2} + q^{-3/2} K_1^{-1} K_2^{-2} \otimes B_1^+ \\ + (q - q^{-1}) q^{-1/2} B_2^+ K_1^{-1} K_2^{-1} \otimes \{B_2^-, B_1^+\}_{q^{-2}} K_2^{-1} \tag{72a}$$

$$\Delta(B_1^-) = q^{3/2} B_1^- \otimes K_1 K_2^2 + q^{1/2} K_1^{-1} K_2^2 \otimes B_1^- \\ + (q^{-1} - q) q^{1/2} \{B_1^-, B_2^+\}_{q^2} K_2 \otimes B_2^- K_1 K_2 \tag{72b}$$

$$\Delta(B_2^\xi) = q^{1/2} B_2^\xi \otimes K_2 + q^{-1/2} K_2^{-1} \otimes B_2^\xi \qquad \xi = \pm \tag{72c}$$

$$\Delta(K_i^\xi) = q^{\xi/2} K_i^\xi \otimes K_i^\xi \qquad i = 1, 2 \qquad \xi = \pm 1. \tag{72d}$$

The morphism π of $U_q[osp(1/4)]$ onto $W(0/2)$ is easily expressed on the deformed pB operators ($k_i \equiv k_i(1)$):

$$\pi(B_i^\pm) = b_i^\pm \qquad \pi(K_i) = k_i \equiv q^{N_i} \qquad i = 1, 2. \tag{73}$$

In order to obtain Green's ansatz of order two (see (51)), it remains to apply the operator $\pi \otimes \pi$ on the right-hand side of expressions (72). Following (32), we generalize the standard Green ansatz notation:

$$b_i^{\pm(1)} = b_i^\pm \otimes e \qquad b_i^{\pm(2)} = e \otimes b_i^\pm \\ k_i^{\pm(1)} = q^{\pm N^{(1)}} = k_i^\pm \otimes e \qquad k_i^{\pm(2)} = q^{\pm N^{(2)}} = e \otimes k_i^\pm \qquad i = 1, 2. \tag{74}$$

In equation (74), e is the unity in $W(0/2)$, which, considered as an operator, is the unit operator $e = id$. From the definition of the (graded) tensor product, it follows that (34) holds, namely

$$\{b_i^{\pm(1)}, b_j^{\pm(2)}\} = 0. \tag{75}$$

Moreover, the operators $k_i^{\pm(1)}$ commute with $b_j^{\pm(2)}$, $k_i^{\pm(2)}$ and $k_i^{\pm(2)}$ commute with $b_j^{\pm(1)}$, $k_i^{\pm(1)}$ for any $i, j = 1, 2$.

Applying $\pi \otimes \pi$ to the right-hand side of (72), we obtain, for the q-deformed Green ansatz operators, the following expressions:

$$b_1^+(2) = b_1^{+(1)} q^{N_1^{(2)} - 2N_2^{(2)} - 1/2} + q^{-N_1^{(1)} - 2N_2^{(1)} - 3/2} b_1^{+(2)} + (q^{1/2} - q^{-7/2}) b_2^{+(1)} q^{-N_1^{(1)} - N_2^{(1)}} b_1^{+(2)} b_2^{-(2)} q^{-N_2^{(2)}} \tag{76a}$$

$$b_1^-(2) = b_1^{-(1)} q^{N_1^{(2)} + 2N_2^{(2)} + 3/2} + q^{-N_1^{(1)} + 2N_2^{(1)} + 1/2} b_1^{-(2)} + (q^{-1/2} - q^{7/2}) b_1^{-(1)} b_2^{+(1)} q^{N_2^{(1)}} b_2^{-(2)} q^{N_1^{(2)} + N_2^{(2)}} \tag{76b}$$

$$b_2^{\xi}(2)_q = b_2^{\xi(1)} q^{N_2^{(2)} + 1/2} + q^{-N_2^{(1)} - 1/2} b_2^{\xi(2)} \qquad \xi = \pm. \tag{77}$$

At $q = 1$, the above expressions reduce to the $p = 2$ relations (1) for two pairs of pB operators.

The example above indicates that the structure of the deformed Green ansatz is more involved. There is, in particular, a big asymmetry between the first pair of operators $b_1^{\pm}(2)_q$ and the second pair $b_2^{\pm}(2)_q$. As a result, the problem of decomposing the tensor product of two Fock representations into a direct sum of irreducible representations of $osp(1/4)$ [41] becomes very difficult in the deformed case. So far we have not been able to solve it.

The asymmetry that appears in (76) and (77) is a consequence of the very different expressions for the comultiplication acting on different pairs of deformed pB operators (see equations (72)). The latter have been derived from the quite symmetrical expressions (69) for the comultiplication defined on the Chevalley generators. We believe it will be possible to write down new symmetric expressions for the comultiplication and, hence, for the deformed Lie-super triple generators (5). To this end, one has to perhaps use multiparametric deformations of $U[osp(2n+1/2m)]$, as indicated in [42]. The requirement that $U_q[osp(2n+1/2m)]$ is a Hopf algebra is also unnecessarily strong. For our considerations, it is sufficient that $U_q[osp(2n+1/2m)]$ is a coalgebra or even less, namely, that there exists a comultiplication which is an algebra morphism, but this is certainly another open problem.

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References

[1] Green H S 1953 *Phys. Rev.* **90** 370
[2] Greenberg O W and Messiah A M L 1965 *Phys. Rev.* B **138** 1155
[3] Ignatiev A Yu and Kuzmin V A 1987 *Yad. Fys.* **46** 486 (Engl. transl. 1987 *Sov. J. Nucl. Phys.* **46** 444)
Greenberg O W and Mohapatra R N 1987 *Phys. Rev. Lett.* **59** 2507
[4] Floreanini R and Vinet L 1990 *J. Phys. A: Math. Gen.* **23** L1019
[5] Celeghini E, Palev T D and Tarlini M 1990 *Preprint* YITP/K-865 Kyoto; 1991 *Mod. Phys. Lett.* B **5** 187

[6] Odaka K, Kishı T and Kamefuchi S 1991 *J. Phys. A: Math. Gen.* **24** L591
[7] Chakrabarti R and Jagannathan R 1991 *J. Phys. A: Math. Gen.* **24** L711
[8] Palev T D 1993 *J. Phys. A: Math. Gen.* **26** L1111
[9] Palev T D 1993 *Lett. Math. Phys.* **28** 321
[10] Hadjiivanov L K 1993 *J. Math. Phys.* **34** 5476
[11] Bonatsos D and Daskaloyannis C 1993 *Phys. Lett.* **307B** 100
[12] Meljanac S, Milekovic M and Pallua S 1994 Unified view of deformed single-mode oscillator algebras *Preprint* RBI-TH-3/94
[13] Flato M, Hadjiivanov L K and Todorov I T 1993 *Found. Phys.* **23** 571
[14] Brzezinski T, Egusquiza I L and Macfarlane A J 1993 *Phys. Lett.* **311B** 202
[15] Macfarlane A J 1993 Generalized oscillator systems and their parabosonic interpretation *Preprint* DAMPT 93-37
[16] Macfarlane A J 1994 *J. Math. Phys.* **35** 1054
[17] Cho K H, Chaiho Rim, Soh D S and Park S U 1994 *J. Phys. A: Math. Gen.* **27** 2811
[18] Chakrabarti R and Jagannathan R 1994 *J. Phys. A. Math. Gen.* **27** L277
[19] Beckers J and Debergh N 1991 *J. Phys. A: Math. Gen.* **24** L1277
[20] Kamefuchi S and Takahashi Y 1960 *Nucl. Phys.* **36** 177
[21] Ryan C and Sudarshan E C G 1963 *Nucl. Phys.* **47** 207
[22] Omote M, Ohnuki Y and Kamefuchi S 1976 *Prog. Theor. Phys.* **56** 1948
[23] Ganchev A and Palev T D 1978 *Preprint* JINR P2-11941; 1980 *J. Math. Phys.* **21** 797
[24] Kac V G 1977 *Adv. Math.* **26** 8
[25] Okubo S 1994 *J. Math. Phys.* **35** 2785
[26] Palev T D 1982 *J. Math. Phys.* **23** 1100
[27] Gel'fand I M and Zetlin M L 1950 *Dokl. Akad. Nauk* **71** 1071
[28] Palev T D 1975 *Ann. Inst. H. Poincaré* **XXIII** 49
[29] Kac V G 1978 *Springer Lecture Notes in Mathematics 626* (Berlin: Springer) p 597
[30] Ky Nguyen Ahn, Palev T D and Stoilova N I 1982 *J. Math. Phys.* **33** 1841
[31] Scheunert M 1979 *Springer Lecture Notes in Mathematics 716* (Berlin: Springer)
[32] Floreaninı R, Spiridonov V P and Vinet L 1991 *Commun. Math. Phys.* **137** 149
[33] Palev T D 1993 *J. Math. Phys.* **34** 4872
[34] Biedenharn L C 1989 *J. Phys. A: Math. Gen.* **22** L873
[35] Macfarlane A J 1989 *J. Phys. A: Math. Gen.* **22** 4581
[36] Sun C P and Fu H C 1989 *J. Phys. A: Math. Gen.* **22** L983
[37] Hayashi T 1990 *Commun. Math. Phys.* **127** 129
[38] Palev T D 1994 *Lett. Math. Phys.* **31** 151
[39] Palev T D and Stoilova N I 1993 *Lett. Math. Phys.* **28** 187
[40] Khoroshkin S M and Tolstoy V N 1991 *Commun. Math. Phys.* **141** 599
[41] Mack G and Todorov I T 1969 *J. Math. Phys.* **10** 2078
[42] Reshetikhin N 1990 *Lett. Math. Phys.* **20** 331